Stroppel

Rep. theory of Lie algebras

In this talk: $\mathfrak{g}$ is a f.d. complex Lie algebra

simple — solvable

Cartan–Killing — wild

semisimple

$$\left\{\begin{matrix}\text{s.s. cpx} \\ \text{Lie alg.}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{connected cpt} \\ \text{Lie group} \\ \text{with trivial center}\end{matrix}\right\}$$

$$\mathfrak{sl}(n,\mathbb{C}) = \underbrace{\begin{bmatrix} 0 & & 0 \\ * & \ddots & 0 \end{bmatrix}}_{\mathfrak{n}_-} \oplus \underbrace{\begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}}_{\mathfrak{h}\ \text{Cartan}} \oplus \underbrace{\begin{bmatrix} 0 & \ddots & * \\ 0 & & 0 \end{bmatrix}}_{\mathfrak{n}_+}$$

Borel $= \mathfrak{h} \oplus \mathfrak{n}_+$

finite dim'l rep's

$\{1 \text{ dim rep.}\} \longleftrightarrow (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$

$\mathbb{C}_\lambda \longleftrightarrow \lambda$

$\mathfrak{sl}(2,\mathbb{C})$

$\left\{\begin{matrix}\text{irr.} \\ \text{finite dim} \\ \text{rep}\end{matrix}\right\}/\simeq \overset{1:1}{\longleftrightarrow} \mathbb{Z}_{\geq 0}$

$V \longmapsto \dim V$

Th (Weyl)

$\mathfrak{g}$: s.s. complex Lie alg. $\Rightarrow$ finite dim rep. are semisimple.

Th (?)

$\{\text{finite dim simple rep.}\} \longleftrightarrow \{\text{dominant integral wt}\}$

$V \longmapsto$ maxal weight

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \qquad V_\lambda = \{v \in V \mid h \cdot v = \lambda(h) v \quad \forall h \in \mathfrak{h}\}$$

Arbitrary simple representations

No classification !

Connections to categorification

1) Want to categorify tensor products of f.d. representations (over quantum groups)

e.g. $\mathfrak{sl}_2(\mathbb{C})$

V_n : n-dim. rep.

$$V_{d_1} \otimes V_{d_2} \otimes \cdots \otimes V_{d_r}$$

want – functorial knot invariant

Jones, RT, $\mathfrak{sl}_k$-inv., Kauffman

– should extend Khovanov

want to see tensor structure

e.g. Lusztig's canonical bases

2) Use categorification to understand

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V$$

simple rep. for the reductive part of $\mathfrak{p}$

First Step Bernstein-Gelfand-Gelfand by introducing category $\mathcal{O}$

Simple weight modules

$\lambda \in \mathfrak{h}^*$

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda =: M(\lambda)$$ Verma module of highest weight λ

- $M(\lambda)_\mu \neq 0 \Rightarrow \mu \leqq \lambda$

- $\dim M(\lambda)_\mu$ = Kostant partition function

- $\mathrm{End}_{\mathfrak{g}}(M(\lambda)) \cong \mathbb{C}$
- $\exists_1$ unique max. submodule $\mathrm{rad}\, M(\lambda)$

  $\Rightarrow M(\lambda)/\mathrm{rad}\, M(\lambda) =: L(\lambda)$ simple

$\mathcal{O}$: simple object $= \{L(\lambda)\}$

Problem. Find characters of $L(\lambda)$ (= dim of weight spaces)

easy $\quad \mathrm{rad}\, M(\lambda) \subset M(\lambda) \twoheadrightarrow L(\lambda)$

has composition series with $L(\mu)$'s with $\mu \leqq \lambda$

but we have to know $[M(\lambda) : L(\mu)]$

how often occurs $L(\mu)$ in $M(\lambda)$.

This is difficult.

- Bernstein-Gelfand[2] introduced category $\mathcal{O}$ to answer this question.
- For finite dimensional $L(\lambda)$, they construct a resolution using Verma's

  $\longrightarrow$ Weyl's character formula
- Kazhdan-Lusztig conjecture $[M(\lambda) : L(\mu)] = P_{\lambda\mu}(1)$

  combinatorially defined polymial

  (categorification of the Hecke alg. is behind.)

$\mathcal{O}(\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+)$ = category of $\mathfrak{g}$-module M s.t.

- finitely generated as $U(\mathfrak{g})$-module
- locally finite w.r.t. $U(\mathfrak{b})$
- $M=\bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda$ weight sp. decomposition

simple objects $L(\lambda)$ $\quad \lambda\in\mathfrak{h}^*$

blocks

$Z\subset U(\mathfrak{g})$ $\quad$ Harish-Chandra $\quad Z \xrightarrow{\sim} U(\mathfrak{h})^W$ polynomial ring

$$\therefore\ \max Z \overset{1:1}{\longleftrightarrow} \mathfrak{h}^*/W$$

$$\mathcal{O}(\mathfrak{g})=\bigoplus_{\chi\in\max Z}\mathcal{O}(\mathfrak{g})_\chi \quad \leftarrow \quad M\in\mathcal{O} \quad \chi^n M=0 \quad n\gg 0$$

finitely many simples in each $\mathcal{O}(\mathfrak{g})_\chi$

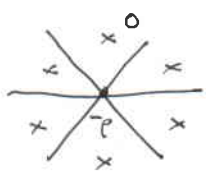
0 = Annihilator of the trivial module

Z

$\mathcal{O}(\mathfrak{g})_0$: "principal block"

contains exactly the simples

$$L(x(0+\rho)-\rho) \quad x\in W$$



ρ = half-sum of positive roots

$\mathcal{O}(\mathfrak{g})_\chi$ abelian, enough proj.

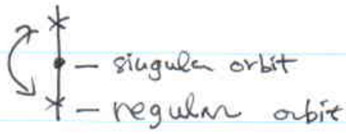
finite dim Hom space , pdim $< \infty$

$\mathcal{O}(\mathfrak{g})_\chi \cong \mathrm{mod}\, A$ $\qquad$ A : finite dimensional algebra

What is A ?

$W = \mathbb{Z}/2$

$\mathfrak{g} = \mathfrak{sl}_2$



$\mathcal{O}(\mathfrak{g})_{\text{sing.}} \cong \mathbb{C}\text{-mod}$ $\qquad$ (only 1 simple)

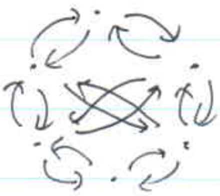
$\mathcal{O}(\mathfrak{g})_{\text{reg.}} \cong \mathrm{Mod}\, A$ $\qquad$ A = path alg. of $1 \underset{g}{\overset{f}{\rightleftarrows}} 2$ $\quad gf = 0$

e_1, e_2, f, g, fg

$\Longrightarrow$ 5-dim. algebra

$\mathrm{End}(P(2)) \cong \mathbb{C}[X]/(X^2) = H_1$

$\mathfrak{sl}_3$



+ relation

A is not explicit in general

Consider $\mathcal{O}(\mathfrak{g})_0$. $\qquad$ $L(x) = L(x(0+\rho)-\rho)$ $\quad x \in W$

simple

$M(0) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}$ is projective in $\mathcal{O}(\mathfrak{g})_0$

- any other proj is a direct summand of
$M(0) \otimes E$ E: finite dim $\mathfrak{g}$-module

$$M(0) \otimes E = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \underline{(\mathbb{C} \otimes E)}$$

filtered   1-dim quotient

$\therefore$ $M(0) \otimes E$ has a filtration with subquotients Verma modules

$P(x)$: projective cover $L(x)$

BGG reciprocity : $[P(x) : M(y)] = [M(y) : L(x)]$

$\Rightarrow$ we "only" have to understand the functors $\cdot \otimes E$ and direct summand

"<u>projective functors</u>"

<u>KL-theory</u> $\left\{ \begin{array}{c} \text{Grothendieck ring of} \\ \text{projective functors} \\ \mathcal{O}(\mathfrak{g})_0 \circlearrowleft \end{array} \right\} \cong \mathbb{Z}W$

"graded version" $\cong$ Hecke alg. of W

Amazing fact : KL can only be proved using geometry!

Example

$$
\begin{array}{lccccc}
\mathfrak{sl}_2 & \mathcal{O}(\mathfrak{g})_{\text{sing.}} & \oplus & \mathcal{O}(\mathfrak{g})_{\text{reg.}} & \oplus & \mathcal{O}(\mathfrak{g})_{\text{sing.}} \\
 & \text{mod-}\mathbb{C} & \oplus & \text{mod-}A & \oplus & \text{mod-}\mathbb{C}
\end{array}
$$

$$
\begin{array}{c} \rotatebox{-90}{$\curvearrowright$} \\ K_0 \end{array} \quad V \otimes V \qquad V \cong \mathbb{C}^2 \circlearrowleft \mathfrak{sl}_2
$$

$$
\begin{array}{ccc}
\cup & \cup & \cup \\
\text{mod } \mathbb{C} & \text{mod } \mathrm{End}\, P(2) & \text{mod } \mathbb{C} \\
 & \shortparallel & \\
 & \text{mod } \mathbb{C}[x]/(x^2) &
\end{array}
$$

$$
\begin{array}{c} \rotatebox{-90}{$\curvearrowright$} \\ K_0 \end{array} \quad V_3 \ : 3\text{-dim.}
$$

$E = \otimes$ natural rep.

$F = \otimes$ natural rep*